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In studying the relation of the axiom of the plane to the supposition of the sum of the angles of the rectilinear triangle being less than 180°, it becomes important to decide whether a certain line which presents itself in our supposed plane is to be considered a straight line or not.

"Der Begriff der Geraden ist vermöge seiner Einfachheit nicht definirbar, Richtung ohne die Gerade unverständlich." Baltzer, Elemente der Mathematik, zweiter Band, Leipzig, 1874. Legendre alludes to the difficulty of defining the straight line. See Erdmann, Axiome der Geometrie, Leipzig, 1877, p. 19.

Suppose in a plane a given straight line; and in the same plane a line defined by its relations to the given straight line. It may be important to decide whether the line so defined is to be considered straight or not.

In the absence of a satisfactory definition of the straight line it is proposed to exclude directly from the category of straight lines by a postulate certain lines in a plane, as being, according to the postulate, forms of deviation from the ideal straight line.

A necessary property of the latter shall be the ideal property of it, that the straight line of any two of its points coincides with that of any other two. This property however leaves the straight line undefined, for a similar relation of any two of the points to their line would hold of the points of a given circular arc, with fixed centre, in a plane.

In the Figure which represents straight lines in a plane, LP, M_2P_2 and L_1P_3 are each a straight line perpendicular to straight line PP_3.

$$LP > M_2P_2$$

$$L_1P_3 > M_2P_2$$

$$QP = M_2P_2 = HP_3$$

$$\alpha = \text{ang. } M_2\ Q\ P$$

$$\beta = \text{ang. } Q\ M_2\ P_2$$

$$\gamma = \text{ang. } M_2\ L\ Q$$

Supposing we are dealing with a plane, and, the sum of the angles of the rectilinear triangle not being greater than 180°, we have

$$\alpha = \beta$$

$$\gamma < 90°$$

Thus in the Figure, the straight lines $L\ M_2$ and $P\ P_3$ are subject to the fifth postulate, as the Figure represents straight lines in a plane. As this postulate concerns straight lines, it cannot, if true, be inconsistent with any other true proposition about such lines.

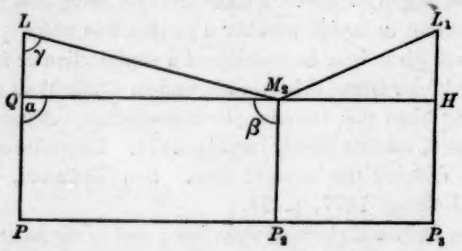

FIGURE.

As to the question of line $L\ M_2\ L_1$ in the Figure being straight or not, besides what may be deduced from the fifth postulate, use may also be made of the property mentioned above. In accordance with this, the straight line of $L\ M_2$ should coincide with that of $M_2\ L_1$, if $L\ M_2\ L_1$ is straight.

But in the Figure, straight-line portion $L\ M_2$ may be considered as converging towards $P\ P_3$, and straight-line portion $M_2\ L_1$ as diverging from $P\ P_3$, the two lines having point M_2 in common, and being on different sides of the perpendicular $M_2\ P_2$, and on the same side of $P\ P_3$.

Now the straight-line portions $L\ M_2$ and $M_2\ L_1$ being conditioned as above described, the postulate may be stated: Straight line passing through points L and M_2 cannot coincide with straight line passing through points M_2 and L_1. The condition stated by the postulate excludes the line $L\ M_2\ L_1$ from being straight, and the postulate may be designated as postulate for a certain form of deviation from the straight line in a plane, or briefly as deviation-postulate.

The ground of this postulate may be shown by means of the following subsidiary postulate. The plane is here regarded as a perfectly flat surface. Consider the straight line in which the limited straight line $L M_2$ lies; and in this straight line, outside the portion $L M_2$ and adjoining M_2 an additional small portion of it, small enough to lie on the same side of $P P_3$ as M_2 does. Now the subsidiary postulate: The additional small straight-line portion commencing at M_2 will converge towards $P P_3$ as $L M_2$ so converges. Consequently $M_2 L_1$, diverging from $P P_3$, must be in a different straight line from that in which $L M_2$ lies, as they both pass through point M_2, and in the respective portions near to M_2 the one converging to, and the other diverging from, $P P_3$, and therefore by the ideal property mentioned above, line $L M_2 L_1$ is not straight.

It may be noted that the convergence of $L M_2$ towards $P P_3$ depends on the given conditions of the construction of the diagram; and in using this convergence in the reasoning it is not necessary to consider whether the straight line of $L M_2$, when prolonged, meets $P P_3$ or not. That is, the reasoning of the subsidiary postulate does not make use of the relation of the fifth postulate which holds valid between the lines $L M_2$ and $P P_3$.

Comparison of the Deviation-Postulate with the Fifth Postulate as They are Indicated on the Figure.

In using the former to show that $L M_2 L_1$ is not straight, it is not necessary to consider whether the course of the straight line $L M_2$ meets $P P_3$ or not; but only that the two limited straight lines $L M_2$ and $M_2 L_1$ being differently inclined to $P P_3$, the one converging towards and the other diverging from it, then, according to the postulate, the straight lines of $L M_2$ and $M_2 L_1$ cannot coincide. This reasoning, it is submitted, does not make use of the fifth postulate.

That $L M_2 L_1$ is not straight may also be considered as a consequence of the fifth postulate, because by that postulate the straight line $L M_2$ would have to meet $P P_3$; but the line $L M_2 L_1$ not doing so, cannot be straight. Here we take into account whether the line $L M_2 L_1$ meets $P P_3$ or not, which was not the case in the preceding reasoning.

That the two postulates which we are comparing are not identical will appear from the following:

The fifth postulate excludes $L M_2$ from taking the course $M_2 H$, which, as $M_2 P_2$ equals $H P_3$, can never meet $P P_3$; but the other postulate considers only those continuations of $L M_2$ which, like $M_2 L_1$, diverge from $P P_3$, and disregards course of $M_2 H$.

It is one proposition to say, as the deviation-postulate does, that straight line $L M_2$ cannot take the course $L M_2 L_1$ as described, thereby excluding $M_2 L_1$ from being the continuation of $L M_2$, and another to say, with the fifth postulate, that the same line must meet $P P_3$, for in this case both $M_2 L_1$ and $M_2 H$ are excluded from being the continuation of $L M_2$.

The two postulates exclude the line $L M_2 L_1$ (see Figure) from being straight, but respectively on different grounds.

The deviation-postulate, considering also the ideal property referred to above as belonging to a straight line, excludes $L M_2 L_1$ from being straight on the ground of its being a course inconsistent with a generic property of the straight line.

The fifth postulate makes the exclusion, because $L M_2 L_1$ being so conditioned at its intersection with $L P$ that $L M_2$ and $P P_3$ are subject to the fifth postulate, does not take the course which, according to this postulate, it should take, or because it does not take the course of one of two straight lines related to each other in a certain way.

Now the line $L M_2 L_1$, considered as a part of diagram (see Figure), embodies both the two sets of conditions which the two postulates respectively present for the line not being straight. It is submitted that either of the two sets may be used for arguing that the line is not straight.

The deviation-postulate may be stated as follows:

Given in a plane two straight-line segments, having a common point outside of a given third straight line in the same plane, and situated on different sides of the perpendicular from the common point to the third straight line; let each of the two straight-line segments, considered as proceeding each from the common point, diverge from the third straight line, then postulate: The two straight lines of the segments do not coincide.

The above postulate is in one respect interesting in comparison with the fifth postulate, in that the former concerns special conditions which prevent a *single* line in a plane from being straight; whereas the latter tells us that *two straight lines* in a plane, and related to each other in a certain way, intersect.

An Additional Difference Between the Two Postulates.

The deviation-postulate is a postulate about two straight lines, and states a condition which prevents a certain line from being straight (see the Figure). Given two adjoining straight-line segments $L M_2$ and $M_2 L_1$,

having a given point in common and related in a certain way, as indicated in description of the Figure, to a given third straight line PP_3, which the segments do not meet. The point in common excludes parallels.

With the given relation to each other that the two segments have a given point in common, a possible case for the two straight lines of these segments would be that they coincide. But, according to the postulate, the relations of the two straight lines to the given third straight line are such that they do not coincide. Then the condition of these two lines is that they intersect at one point. The part of this condition which the postulate of the two lines states, is

The exclusion of coinciding.

Fifth postulate: With the relation to each other that the two straight lines of the postulate do not coincide, and are related in a certain way to a third straight line which they both intersect; then postulate. The two straight lines of the postulate intersect. This, the intersection, is equivalent to

Excluding parallels.

Therefore, the postulate in this case virtually states the exclusion of parallels, whereas, in the case of the deviation-postulate, it states exclusion of coinciding.

Two uses may be made of the above deviation-postulate: one, to show that the sum of the angles of the rectilinear triangle is equal to 180°; and the other, to show that if that sum is supposed less than 180° then the axiom of the plane does not hold good.